Reminders:
HW 2 assigned (see Canvas)
1-2 page project proposal (see Canvas)

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Today:
- Proof that interval indecomposables characterize type A quiver reps up to iso.
- interleavings
- interleaving / bottleneck distance isometry
- examples
- Posets

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alg to put quiver rep in "barcode form"

Thm: barcode form uniquely determines iso. class.

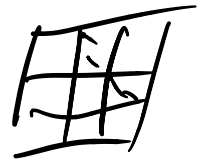
Why do we care? Want to know barcodes don't depend on arbitrary choices.

Prop: Every finite dim. quiver rep of type $A_n$ has a barcode form.

Pf: follows from alg. existence LEUP fact.

Prop: 

Barcode fact: $A = B \Lambda B^{-1}$ is a quiver rep. iso.



Pf: $\underbrace{L}E\underbrace{UP}$ $L, U, P$ all invertible, so all steps involved invertible change of basis.

$I(c,d) \oplus I[a,b]$

$I[a,b] \oplus I[c,d]$

Pf of thm:

1) Suppose $A, A'$ have same barcode form

i.e. $A = B \Lambda B^{-1}$, $A' = B' \Lambda (B')^{-1}$

$\Rightarrow A = \underbrace{B(B')^{-1}}_{\text{iso}} A' \underbrace{B' B^{-1}}_{\text{iso}}$

$\to A, A'$ are iso. as quiver reps.

2) Suppose two type-$A_n$ quiver reps are iso. ie $A' = MAM^{-1}$ ($M$ invertible, block diag)

The alg. will play out differently

Let $v \in V_b$ be a basis elt. for $I[b,d]$ in $A$, and $v' = Mv$. Note that $v'$ may be a lin. comb. of basis elts in $A'$

$$\begin{tikzcd} v & A_i v & \\ V_i \arrow[r, leftrightarrow, "A_i"] \arrow[d, "M_i"'] \arrow[dr, phantom, "\mathrm{Co}"] & V_{i+1} \arrow[r, leftrightarrow, "A_{i+1}"] \arrow[d, "M_{i+1}"] & {} \\ V'_i \arrow[r, leftrightarrow, "A'_i"'] & V'_{i+1} \arrow[r, leftrightarrow] & {} \\ v' = Mv & A'_i v' & \end{tikzcd}$$

Case 1: $V_i \to V_{i+1}$ : Either $A_i v = 0$ or $A_i v \neq 0$

image $A_i' v'$ must be same

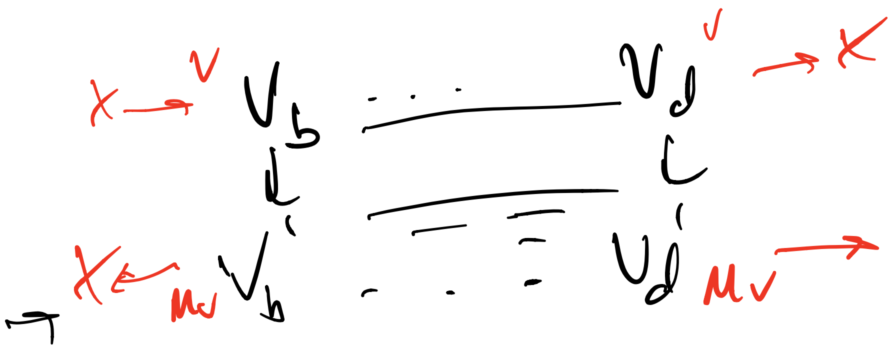
Case: $V_i \leftarrow V_{i+1}$ : Either $v$ is in img $A_i$ or not

same for $v'$ in img $A_i'$

Either case: propagate $v$ from $V_b$ to $V_d$.

follow img or inverse img

$\rightarrow$ $v'$ can propagate from $V_b'$ to $V_d'$



$I[b,d]$

$I[b,d] \oplus I[b_2,d_2]$

$b \leq b_2,\ d_2 \leq d$

If we consider $v'$ in indecomposable basis for $A'$ must be component in $I[b,d]$, has to be "longest elt"

we will associate $v$ with this component

basis elts in $A$ $\rightarrow$ basis elts in $A'$

Now, suppose there is a 2nd basis elt in $A$ ($w$) that gets associated w/ same basis elt in $A'$

$\Rightarrow \exists\ \alpha$ s.t. $M(v - \alpha w)$ has 0-coeff on $I[b,d]$ in $A'$ $\Rightarrow$ $v - \alpha w$ for diagram to commute.

$v, w \quad I[b,d]$

$V_b - - - V_d$

$V'_b - - - V'_d$

$[\longrightarrow]$

$\Rightarrow$ indecomposables of $A$ map injectively to indecomposables of $A'$.

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$M^{-1}$, $\underline{A'} \to \underline{A}$.

use same argument.

$\Rightarrow$ Bijection btw. indecomposables of $A$, $A'$ $\quad I[b,d]$

$\Rightarrow$ barcodes are identical. $\quad I[b,d]$ $\square$

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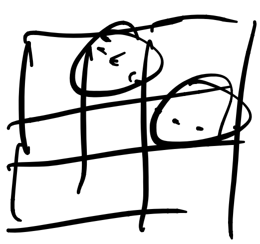
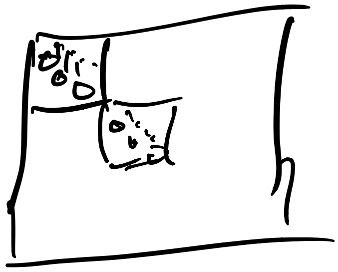
Note: for persistence type quivers:

$\bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet$

Barcode form:



can permute out of block structure to group indecomposables together:



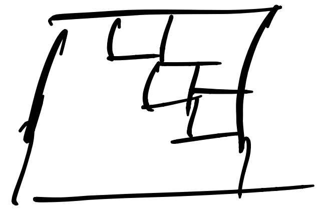
each block look like

$$\begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix} \} \text{ Jordan 0-block.}$$

we can understand barcode form for persistence quiver reps as a permuted version of Jordan form.

A v : Jordan 0-blocks describe

nilpotence : $A^k v = 0$



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one interesting note:

Jordan form generally only exists for algebraically closed fields. ($\mathbb{C}$, $\cancel{\mathbb{R}}$)

$x^2+1=0$

in this case, it always exists (didn't depend on field)

- Henselman 2017, Jordan form always exists for nilpotent operators (only 0-blocks)

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How to deal w/ real-valued filtrations:

$V_{(0)} \to V_{(1)} \to V_{(2)}$

in case where we have a finite # of homological crit. values, we can just look at the quiver rep.

$V_{t_0} \to V_{t_1} \to \dots$

where $\{t_0, t_1, \dots\}$ are critical

interpret indecomposables as $I[t_b, t_d]$

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Interleavings: $\varepsilon$

we can characterize type-A quiver reps up to iso.

what about perturbations on algebraic level.?

Deal w/ quiver reps over $\mathbb{R}$.

define $\varepsilon$-shift map: $F_\varepsilon : Q \to Q'$ as a set of maps $F_\varepsilon : V_t \to V'_{t+\varepsilon}$ that commute

$$\begin{tikzcd} \underset{H(x_s)}{V_s} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \underset{H(x_t)}{V_t} \arrow[d, "F"] \\ V'_{s+\varepsilon} \arrow[r] & V'_{t+\varepsilon} \end{tikzcd} \qquad s \le t$$

define increment map $\iota_\varepsilon : Q \to Q$ is the case which coincides w/ tracing maps in $Q$

$$i_\varepsilon : V_t \xrightarrow{A} V_{t+\varepsilon} \quad \forall t$$

Def: two quiver reps $Q, Q'$ are $\varepsilon$-interleaved if $\exists\ F_\varepsilon : Q \to Q',\ G_\varepsilon : Q' \to Q$ s.t.

$G_\varepsilon \circ F_\varepsilon = i_{2\varepsilon}$ (on $Q$)

$F_\varepsilon \circ G_\varepsilon = i'_{2\varepsilon}$ (on $Q'$)

$i_{2\varepsilon}$

$$
\begin{tikzcd}
V_{t-\varepsilon} \arrow[r] \arrow[dr, "F"] \arrow[rr, bend left] & V_t \arrow[r] \arrow[dr, "F"] & V_{t+\varepsilon} \\
V'_{t-\varepsilon} \arrow[r] \arrow[ur, "G"] \arrow[rr, bend right, "i'_{2\varepsilon}"'] & V'_t \arrow[r] \arrow[ur, "G"] & V'_{t+\varepsilon}
\end{tikzcd}
$$

Def: The interleaving distance btw $Q, Q'$ is $d_I(Q, Q') = \inf\{\varepsilon \geq 0 \mid Q, Q'$ are $\varepsilon$-interleaved$\}$

if $Q \cong Q \Rightarrow d_I = 0$

$V_t \rightleftarrows V'_t$

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Example: let $X$ be a simplicial cpx, $f, g: X_0 \to \mathbb{R}$

Let $X_f$ be lower-star filt. for $f$.

$X_g$ $g$

simplex $(x_0 \ldots x_k)$ appears at

$\rightarrow \max_{i=0..k} f(x_i)$ in $X_f$

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then $d_I(PH_k(X_f), PH_k(X_g)) \leq \|f-g\|_\infty$

pf: let $\varepsilon = \|f-g\|_\infty = \max_{x \in X_0} |f(x)-g(x)|$

$$\begin{cases} \rightarrow g(x_i)-\varepsilon \leq f(x_i) \leq g(x_i)+\varepsilon \quad \forall x_i \in X_0 \\ \rightarrow X_g(t-\varepsilon) \subseteq X_f(t) \subseteq X_g(t+\varepsilon) \quad \forall t \\ X_f(t-\varepsilon) \subseteq X_g(t) \subseteq X_f(t+\varepsilon) \end{cases}$$

Have a diagram of inclusions

$$\begin{tikzcd}
X_f(t-\varepsilon) \arrow[r, phantom, "\subseteq"] \arrow[dr, "\subseteq"] & X_f(t) \arrow[r, phantom, "\subseteq"] \arrow[dr, "\subseteq"] & X_f(t+\varepsilon) \\
X_g(t-\varepsilon) \arrow[r, phantom, "\subseteq"] \arrow[ur, "\subseteq"] & X_g(t) \arrow[r, phantom, "\subseteq"] \arrow[ur, "\subseteq"] & X_g(t+\varepsilon)
\end{tikzcd}$$

Apply Homology functor

$$\begin{tikzcd}
PH(X_f) \quad H_k(X_f(t-\varepsilon)) \arrow[r] \arrow[dr] & H_k(X_f(t)) \arrow[r] \arrow[dr] & H_k(X_f(t+\varepsilon)) \\
H_k(X_g(t-\varepsilon)) \arrow[r] \arrow[ur] & H_k(X_g(t)) \arrow[r] \arrow[ur] & H_k(X_g(t+\varepsilon)) \\
PH(X_g) & &
\end{tikzcd}$$

Commutes b/c all maps induced by inclusion.

$\Rightarrow d_I(PH(X_\varepsilon), PH(X_g)) \leq \varepsilon$

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Note: def of $d_I$ means that $d_I(Q, Q \setminus \Delta) = 0$
where $Q - \Delta$ removes 0-length bars/indecom.

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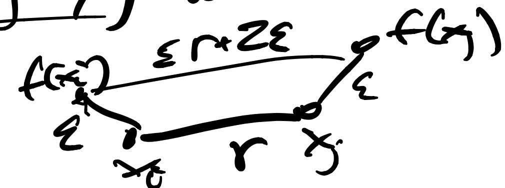
Example 2: Let $d_H(X, Y) = \varepsilon$ $(X, Y)$ pt clouds

$\Rightarrow d_I(PH_k(R(X)), PH_k(R(Y))) \leq 2\varepsilon$.

Pf: $d_H(X, Y) = \varepsilon \Rightarrow \forall x \in X \; \exists y \in Y$ s.t. $d(x,y) \leq \varepsilon$
$\forall y \in Y \; \exists x \in X$ s.t. $d(x,y) \leq \varepsilon$

$\forall x \in X$ choose one $y = f(x)$ within $\varepsilon$

$\forall y \in Y$ choose one $x = g(y)$ within $\varepsilon$




Let $(x_0 \ldots x_k) \in R(X; r)$

$\Rightarrow (f(x_0) \ldots f(x_k)) \in R(Y; r + 2\varepsilon)$

$(y_0 \ldots y_k) \in R(Y; r)$

$\Rightarrow (g(y_0) \ldots g(y_k)) \in R(X; r + 2\varepsilon)$

Simplicial maps $f: R(X; r) \to R(Y; r + 2\varepsilon)$
$g: R(Y; r) \to R(X; r + 2\varepsilon)$

$$\begin{tikzcd}
R(X; r-2\varepsilon) \arrow[r, hook] \arrow[dr, "f"] & R(X; r) \arrow[r, hook] \arrow[dr, "f"] & R(X; r+2\varepsilon) \\
R(Y; r-2\varepsilon) \arrow[r, hook] \arrow[ur, "g"] & R(Y; r) \arrow[r, hook] \arrow[ur, "g"] & R(Y; r+2\varepsilon)
\end{tikzcd}$$

$g \circ f : (x_0 \ldots x_k) \mapsto (g\circ f(x_0) \ldots g \circ f(x_k))$
$\neq (x_0 \ldots x_k)$!

but $g \circ f \simeq i : R(X; r) \to R(X; r+4\varepsilon)$
just construct a retraction from
$r : (x_0 \ldots x_k) \mapsto (g\circ f(x_0), \ldots g\circ f(x_k))$

sim. $f \circ g \simeq i$

pass to homology: b/c homotopic to $i$
induce same map on homology.

$\Rightarrow$ interleaving

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Stability thm for lower-star filtrations
$d_I \leq \| \quad \|_\infty$

Stability thm for Rips complexes
$d_I \leq 2 d_H$

Isometry theorem: $d_I = d_B$ → geometric construction

↳ algebraic construction

How to prove stability: construct an interleaving

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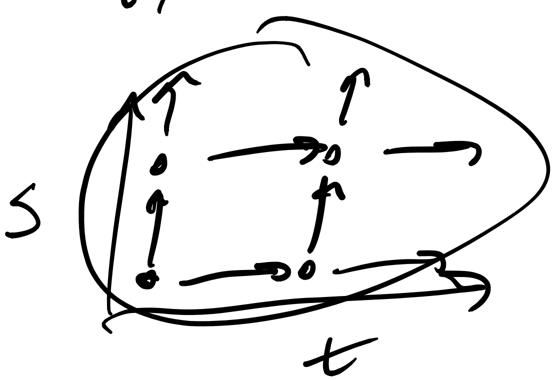
we can construct filtrations.

what about bifiltrations?

example: Rips parameter, density parameter

$X_{t,s} : \{x \mid f(x) \le t,\ g(x) \le s\}$




"multi-dimensional pers." 2008.

One lesson from quiver reps.

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*RIVET. Multi-D persistence.

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No barcodes. $\Rightarrow$ no $d_B$

However, $d_I$