Quiver Representations
Zig Zag Homology

Oudot Chapter 1

Filtrations
$X_0 \subseteq X_1 \subseteq X_2 \ldots$

$\Downarrow$

$H_k(X_0) \to H_k(X_1) \to H_k(X_2) \to \ldots$
persistent homology

Nothing stopping us from
$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \to \ldots$

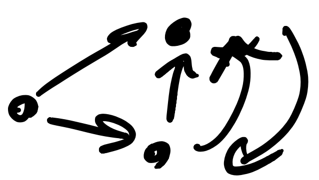
$\Downarrow$

$H_k(X_0) \xrightarrow{\hat{F}_0} H_k(X_1) \xrightarrow{\hat{F}_1} \ldots$

Carlsson, de Silva 2009 "Zig Zag Persistence"

Persistent Homology an example of type-A Quiver representation.

What is a quiver?
In CS: a directed multigraph.



Persistence-type quiver

$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$

Quiver Representation:

Vector space on every node

Linear map for every edge

$$\begin{tikzcd} V_0 \arrow[r, "A_0"] \arrow[d, "A_1"'] & V_1 \\ V_2 \arrow[ur, "A_2"] \arrow[ur, bend right, "A_3"'] & \end{tikzcd}$$

$V_1 \xrightarrow{A_1} V_2 \xrightarrow{A_2} V_3 \cdots$

☺

$H_k(X_1) \xrightarrow{F_1} H_k(X_2) \xrightarrow{F_2}$

Data for quiver rep: set of vector spaces

$Q(V,E)$ $\qquad V_i$

lin. trans. $A_{ij}: V_i \to V_j \;\; \forall (i,j) \in E$

Classification of quiver reps:

Morphism of quiver reps:

$B: Q^1(\underline{V},E) \to Q^2(\underline{V},E)$

- underlying graphs identical $\quad \circ \longrightarrow \bullet \longrightarrow \bullet$ $\quad E$
- $B_i: V_i^1 \to V_i^2 \quad \forall\, i \in V$
-

$$\begin{tikzcd} V_i^1 \arrow[r, "A_{ij}^1"] \arrow[d, "B_i"'] \arrow[dr, phantom, "G_2"] & V_j^1 \arrow[d, "B_j"] \\ V_i^2 \arrow[r, "A_{ij}^2"] & V_j^2 \end{tikzcd}$$

"G" $\quad \circ \longrightarrow \bullet \longrightarrow \bullet$

commutes: $B_j A_{ij}^1 = A_{ij}^2 B_i$

$\forall (i,j) \in E$

Two quiver reps Q^1, Q^2 are isomorphic if $\exists$ $B^1 : Q^1 \to Q^2$, $B^2 : Q^2 \to Q^1$ s.t. $B^2 \circ B^1 = id^1$ $B^1 \circ B^2 = id^2$

Krull-Schmidt: Quiver Reps can be classified as $Q \cong I_0 \oplus I_1 \oplus \cdots$

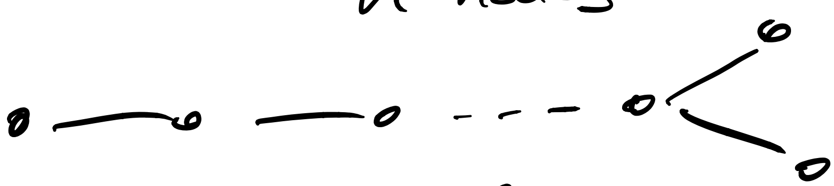
$I_0, I_1, \ldots$ "indecomposables"

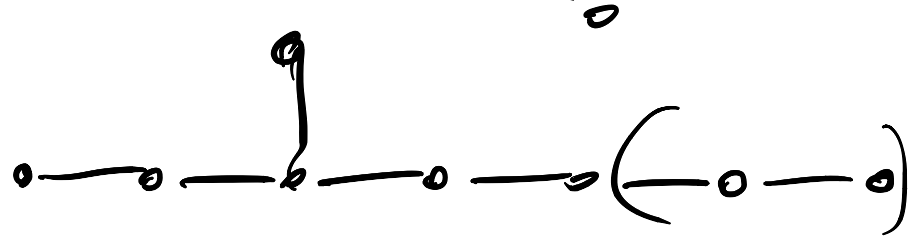
$$I_0 \not\cong I_0' \oplus I_1'$$

Gabriel '72: Only certain types of quiver reps have finitely many iso. classes of indecomposables

- This depends entirely on the underlying graph
- This does not depend on edge direction

Type A_n: $\underbrace{\circ - \circ - \circ \cdots \circ}_{n \text{ nodes}}$

D_n: 

E_6, E_7, E_8 

"Dynkin diagrams" / ADE classification

Important thing: type-A quiver reps encompass P.H.

Carlsson & de Silva '09 arrow directions don't matter, so compute homology on diagrams like

$$X_0 \nearrow X_0 \cup X_1 \nwarrow X_1 \nearrow X_1 \cup X_2 \nwarrow X_2 \nearrow \cdots$$

type quiver rep:

$$H_k(X_0) \longrightarrow H_k(X_0 \cup X_1) \longleftarrow H_k(X_1) \longrightarrow \cdots$$

a "zig zag homology": just like PH, get a barcode. barcode equivalent to indecomposables for type-A quiver reps.

Interval indecomposables

$$\begin{array}{ccccccc} 1 & & 2 & & 3 & & 4 \\ \circ & — & \circ & — & \circ & — & \circ \end{array}$$

$$\begin{array}{ccccccccc} 0 & — & \mathbb{F} & \equiv & \mathbb{F} & \equiv & \mathbb{F} & & I[2,4] \\ 0 & — & \mathbb{F} & \equiv & \mathbb{F} & — & 0 & & I[2,3] \end{array}$$

$I[2,4] \oplus I[2,3]$

$$0 \longrightarrow \mathbb{F}^2 \xrightarrow{\begin{bmatrix}1&0\\0&1\end{bmatrix}} \mathbb{F}^2 \underset{\begin{bmatrix}1&0\end{bmatrix}}{\longrightarrow} \mathbb{F}$$

$$\mathbb{F}^2 \xleftarrow{\begin{bmatrix}1\\0\end{bmatrix}} \mathbb{F}$$

A_4-quiver

barcode: $\{(2,4),(2,3)\} \Leftrightarrow I[2,4] \oplus I[2,3]$

How to compute interval indecomposables?

1) reduction alg.
2) for arbitrary type-A quiver reps?

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indecomposable form

$$\bullet \xrightarrow{A_1} \bullet \xrightarrow{A_2} \bullet \xrightarrow{A_3} \bullet$$

$$\cong$$

$$\bullet \xrightarrow{Q_1} \bullet \xrightarrow{Q_2} \bullet \xrightarrow{Q_3} \bullet$$

Q "pivot matrix"
at most 1-nonzero in each row and column.

$\begin{bmatrix}1&0\\0&0\end{bmatrix}$ $\begin{bmatrix}0&1\\1&0\end{bmatrix}$

$\begin{bmatrix}1&0\\0&1\end{bmatrix}$

$\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}$

read off barcode
if in indecomposable form

for $k = 1 \dots n$

- for $v_j \in V_k$
  - if $V_k \xrightarrow{Q_k} V_{k+1}$ and $Q_k[i,j] \neq 0$
  - or $V_k \xleftarrow{Q_k} V_{k+1}$ and $Q_k[j,i] \neq 0$
    - v_i continues bar represented by v_j
  - else bar with v_j dies
- for $v_i \in V_{k+1}$ not matched w/ $v_j \in V_k$
  - start a new bar rep. v_i

Computations:

Quiver rep. on n vertices $\Rightarrow$ Matrix n blocks. Block non-zero structure same as adjacency matrix of underlying directed graph (quiver)

$$v_0 \xleftarrow{A_1} v_1 \xleftarrow{A_2} v_2$$

$$\begin{array}{c|ccc} & v_0 & v_1 & v_2 \\ \hline v_0 & & A_1 & \\ v_1 & & & A_2 \\ v_2 & & & \end{array}$$

$$\circ \xleftarrow{A_1} \circ \xrightarrow{A_2} \circ$$

$$\begin{array}{|c|c|c|} \hline & A_1 & \\ \hline & & \\ \hline & A_2 & \\ \hline \end{array}$$

block dims = dims of vector spaces for quiver rep. "Companion matrix" $A(Q)$

Quiver iso: • underlying graphs same
• vector space dims same

and $\exists B$ s.t. $A(Q_1) = B^{-1} A(Q_2) B$

B is a block-diagonal change of basis matrix

$$\begin{tikzcd}
Q_1: \bullet \arrow[d, "B_1"'] & \bullet \arrow[l, "A_1^1"'] \arrow[d, "B_2"'] & \bullet \arrow[l, "A_2^1"'] \arrow[d, "B_3"'] \\
Q_2: \bullet \arrow[d, "B_1^{-1}"'] & \bullet \arrow[l, "A_1^2"'] \arrow[d, "B_2^{-1}"'] & \bullet \arrow[l, "A_2^2"'] \arrow[d, "B_3^{-1}"'] \\
\bullet & \bullet \arrow[l, "A_1^1"'] & \bullet \arrow[l, "A_2^1"']
\end{tikzcd}$$

} quiver iso form

$$\begin{bmatrix} & A_1^1 & \\ & & A_2^1 \\ & & \end{bmatrix} = \begin{bmatrix} B_1^{-1} & & \\ & B_2^{-1} & \\ & & B_3^{-1} \end{bmatrix} \begin{bmatrix} & A_1^2 & \\ & & A_2^2 \\ & & \end{bmatrix} \begin{bmatrix} B_1 & & \\ & B_2 & \\ & & B_3 \end{bmatrix}$$

Companion matrix form

Corollary: if B is invertible

$$\underset{v_1}{\bullet} \xleftarrow{A_1 B} \underset{v_2}{\bullet} \xleftarrow{A_2} \underset{v_3}{\bullet} \quad \cong \quad \bullet \xleftarrow{A_1} \bullet \xleftarrow{B A_2} \bullet$$

(change of basis on V_2)

Companion matrix in "barcode form" if all blocks are pivot matrizes

Building Block: LE_LUP factorization

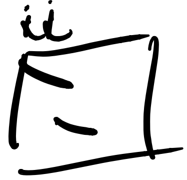
$$A = LE_LUP$$

L lower tri

U upper tri

P perm. matrix

→ E_L "Echelon Lower" (type of pivot matrix)



lower tri.

column pivots in increasing order

zero columns on the right.

Variant of LU fact. w/ column pivoting

Alg: $A \in F^{m \times n}$

$L = I_m$, $U = I_n$, $P = I_n$, $E = A$, $i = 1$, $j = 1$

while $i \le m$, $j \le n$

  if row i in E has non-zero in column $j' \ge j$

    swap columns j, j' in E

    take Schur complement wrt. i,j entry

    i++, j++

  else

    i++

return L, E, U, P

Pf: by induction (i) $A = LEUP$ $\forall i$

✓ for $i = 0$

at step i:

$$A = \begin{bmatrix} L_{11} & 0 & 0 \\ L_{i1} & 1 & 0 \\ L_{21} & 0 & I \end{bmatrix} \begin{bmatrix} E_{11} & 0 & 0 \\ 0 & A_{ij} & A_{i2} \\ 0 & A_{2j} & A_{22} \end{bmatrix} \begin{bmatrix} U_{11} & U_{1j} & U_{12} \\ 0 & 1 & 0 \\ 0 & 0 & I \end{bmatrix} P$$

if no nonzeros in A_{ij} or A_{i2}, increment i

if nonzero:

permutations of columns:

lemma: $\begin{bmatrix} I & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ 0 & I \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12}P^T \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & P \end{bmatrix}$

pf: $\begin{bmatrix} U_{11} & U_{12} \\ & P \end{bmatrix}$ (take products)

Schur complement: Assume $A_{ij} \neq 0$

$$\begin{bmatrix} E_{11} & & \\ & A_{ij} & A_{i2} \\ & A_{2j} & A_{22} \end{bmatrix} =$$

$$\underbrace{\begin{bmatrix} I & 0 & 0 \\ 0 & 1 & 0 \\ 0 & A_{2j}A_{ij}^{-1} & I \end{bmatrix}}_{L} \begin{bmatrix} E_{11} & 0 & 0 \\ 0 & A_{ij} & 0 \\ 0 & 0 & S \end{bmatrix} \underbrace{\begin{bmatrix} I & 0 & 0 \\ 0 & 1 & A_{ij}^{-1}A_{i2} \\ 0 & 0 & I \end{bmatrix}}_{U}$$

$$S = A_{22} - A_{2j}A_{ij}^{-1}A_{i2}$$

Pass left & right terms to L & U

Lower tri matrices are closed under mult.
upper " " " " " "

$\square$

Notes: L, U, P are invertible, E_L not generally

Corollary 1: $L' \overset{\text{not invertible}}{L} = L E_L$ (b/c U, P are identity)

Corollary 2: $E_L L = \tilde{L} E_L$

Pf: $E_L L$ is lower-triangular
apply cor. 1

Algorithm for barcode form of persistence quiver rep

$$\circ \xleftarrow{L^1 E_L^1} \circ \xleftarrow{L^2 E_L^2 \, UP} \circ \; - - - \; \xleftarrow{A_{n-1}} \circ$$

1) $A_1 = L E_L UP$
2) Pass U, P to A_2, $UPA_2 = \tilde{A_2}$
3) $\tilde{A_2} = L E_L UP$
4) ...

$\bullet \xleftarrow{L^1 E_L^1} \bullet \xleftarrow{L^2 E_L^2} \cdots \xleftarrow{L^{n-1} E_L^{n-1}} \bullet$

$\bullet \xleftarrow{L^1 E_L^1} \bullet \xleftarrow{L^2 E_L^2 L^3} \bullet \cdots \xleftarrow{E_L^{n-1}}$

$\bullet \xleftarrow{L^{2'} L^3 E_L^2} \bullet \cdots$

$\xleftarrow{\quad} L^{2'}$

$\bullet \xleftarrow{L^{1'} E_L^1 L^{2'}} \bullet \xleftarrow{E_L^2} \bullet \cdots \xleftarrow{E_L^{n-1}}$

$\bullet \xleftarrow{E_L^1} \bullet \xleftarrow{E_L^2} \bullet \cdots \xleftarrow{E_L^{n-1}}$ in barcode form ✓

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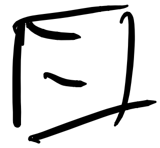
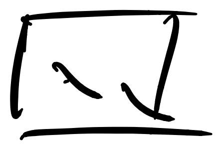
what abt zigzags? $\rightarrow$ $\leftarrow$

A: need $PU\hat{E}_L L$ factorization

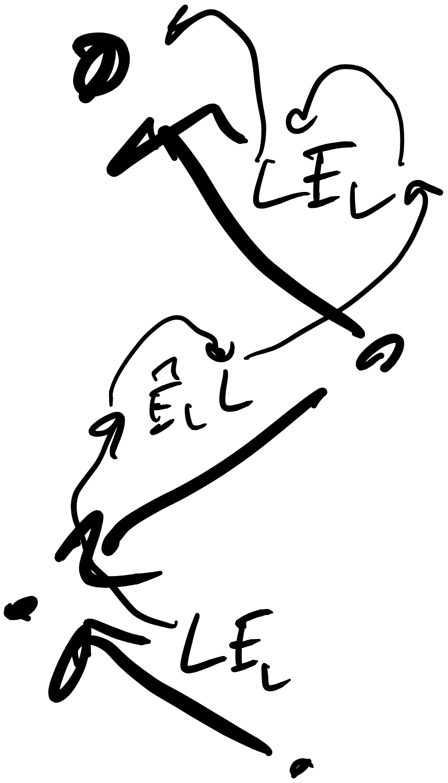
E_L  $\hat{E}_L$ 

Exercise: derive from LE_LUP (modify index order in loops)

Cor: $L\hat{E}_L = \hat{E}_L \hat{L}$

$\bullet \xleftarrow{A_1} \bullet \xrightarrow{A_2} \bullet \xleftarrow{A_3} \bullet$

$\Downarrow$



$\Rightarrow$ Barcode form of Zigzag

Ref: "Zigzag homology" by Carlsson & de Silva

"Persistent & Zigzag homology: A matrix factorization viewpoint"
C. O. N.