Homotopy
Discrete Morse Theory ← simplification
Schur Complements

Recall homotopy: $f, g : X \to Y$

$$f \simeq g \iff \exists\, h : X \times I \to Y$$
$$h(\cdot, 0) = f$$
$$h(\cdot, 1) = g$$

In chain category, notion of homotopy.

Chain maps $F_*, G_* : C_* \longrightarrow D_*$ $\left(F_{k-1} \partial^C_k = \partial^D_k F_k\right)$

Chain homotopy H_*: $H_k : C_k \longrightarrow D_{k+1}$

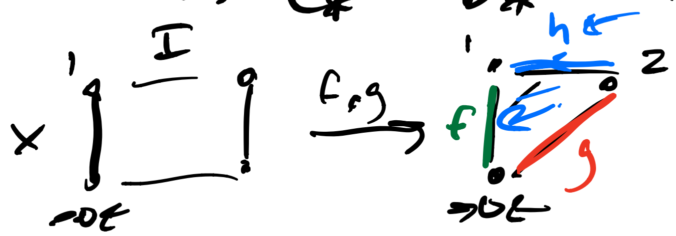
$\to \partial^D_{k+1} H_k + H_{k-1} \partial^C_k = G_k - F_k \implies F_* \simeq G_*$

We say C_* and D_* are homotopy equiv

if $\exists\ F_* : C_* \to D_*$ and $G_* : D_* \to C_*$

s.t. $G \circ F \simeq id^C$, $F \circ G \simeq id^D$

$\implies C_* \simeq D_*$



F: $(0) \to (0)$
$(1) \to (1)$
$(0,1) \to (0,1)$

G: $(0) \to (0)$
$(1) \to (2)$
$(0,1) \to (0,2)$

H: $(0) \mapsto (0,0) = 0$
$(1) \mapsto (1,2)$
$(0,1) \mapsto -(0,1,2)$

in dim 0:

$$\partial H_0 + \underset{\sim}{H_{-1}}\partial = G - F$$

(0): $0 + 0 = (0) - (0) = 0$ ✓

(1): $\partial(1,2) + 0 = (2) - (1) = G(1) - F(1)$ ✓

in dim 1:

$$\partial H_1 + H_0\partial = G - F?$$

$$-\partial(0,1,2) + H_0((1) - \underline{(0)}) = -(0,1) - \cancel{(1,2)} + (0,2) + \cancel{(1,2)} + 0$$

$$= \underbrace{-(0,1)}_{-F(0,1)} + \underbrace{(0,2)}_{G(0,1)} \checkmark$$

Prop: if $F_* \simeq G_*$, then $H_k(F_*) = \tilde{F}_k \simeq \underset{\tilde{G}_k}{H_k(G_*)}$

(homotopic maps induce same map on homology)

Pf: Let $F_*, G_* : C_* \to D_*$, $F_* \simeq G_*$

let $[x] \in H_k(C_*)$ we want to show

$$[F_k x] = [G_k x]$$

ie $G_k x - F_k x \in \operatorname{img} \partial^D_{k+1}$

because $G_* \simeq F_*$ $\exists$ H_* s.t.

$G_k x - F_k x = \underbrace{\partial_{k+1} H_k x}_{\in \operatorname{img} \partial_{k+1}} + \underbrace{H_k \partial_k x}_{\substack{=0 \text{ b/ } x \in \ker \partial_k \\ \text{(homology rep)}}}$

✓

$\tilde{F}_k, \hat{G}_k$ send homology reps to same homology class → induced maps are same. ☐

Corollary: if $C_* \simeq D_*$ then $H_k(C_*) \cong H_k(D_*)$

Homotopy invariance of homology:

$X \simeq Y \Rightarrow C_*(X) \simeq C_*(Y) \Rightarrow H_k(X) \cong H_k(Y)$

$f \simeq g \Rightarrow F_* \simeq G_* \Rightarrow \tilde{F}_* \cong \hat{G}_*$

Discrete Morse Theory:

Morse theory connects smooth manifolds w/ cell complexes. Big subject in math.

We'll look at how discrete Morse theory can simplify complexes, ultimately homology calculations.

Ref: Mischaikow, Nanda 2013 ←

Def: Let X be a (simplicial) complex. An incidence function $\kappa: X \times X \to R$ pairs simplices w/ codimension-1 faces

$\kappa(\sigma, \tau) \neq 0$ if τ is a codim-1 face of σ.

note: ∂ in chain cpx is an incidence

The face partial order $\leq$ on X is generated by the relation $<$ $\tau < \sigma$ if $\kappa(\sigma, \tau) \neq 0$

$\rho \leq \sigma$ if $\exists\ \tau_0 \ldots \tau_n$ $\rho < \tau_0 < \tau_1 \ldots < \tau_n < \sigma$

Def: a partial matching of (X, κ) partitions X into 3 sets: A, K, Q, with bijection $w: Q \to K$ s.t. $\forall\ q \in Q$, $\kappa(w(q), q) \neq 0$

denote $(A, w: Q \to K)$

$\Rightarrow \dim w(q) = \dim q + 1$ $\quad q < w(q)$

<u>matching</u> is <u>acyclic</u> b/c w obeys partial order

A gradient path in an acyclic matching: is a sequence of cells $\rho: (q_1, w(q_1), \ldots q_m, w(q_m))$ with $q \in Q$ s.t. $q_i \neq w(q_{i+1}) < w(q_i)$

$(0) < (0,1)$

$(1,2) < (0,1,2)$

$((1,2), (0,1,2), (0), (0,1))$

The index $\nu(\rho)$ is defined

$$\nu(\rho) = \frac{\prod_{i=1}^{m-1} \kappa(w(q_i), q_{i+1})}{\prod_{i=1}^{m} -\kappa(w(q_i), q_i))}$$

initial cell q_1 of ρ denoted $q_\rho \in Q$

final cell $w(q_m)$ . . . $\kappa_\rho \in K$

given cells $a, a' \in A$, a gradient path φ is a connection from a to a' if

$q_\varphi < a, \quad a' < k_\varphi \;:\; a \xrightarrow{\varphi} a'$

The multiplicity of the connection φ is defined

$m(\varphi) : \kappa(a, q_\varphi) \cdot \nu(\varphi) \cdot \kappa(k_\varphi, a')$

define a new map $\tilde{\kappa} : A \times A \to R$ by the relation $\tilde{\kappa}(a, a') = \kappa(a, a') + \sum_{\varphi : a \to a'} m(\varphi)$

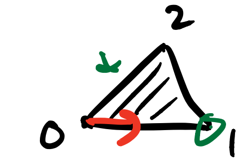
(0 if no connecting φ)

A complex $(A, \tilde{\kappa})$ is called the <u>Morse</u> complex associated to the acyclic matching $(A, w : Q \to k)$ of X, and $\tilde{\kappa}$ is called the Morse incidence function.

Thm: if we consider A as a restriction of X to unmatched cells (simplices) and take $\tilde{\kappa}$ as boundary map of A, then

$H_*(X) \cong H_*(A)$

Example:



$Q = \{(0)\}$.

$K = (0,1)\}$

$A = \{(1), (2), (1,2), (0,2), (0,1,2)\}$

$w: (0) \mapsto (0,1)$

$\varphi = ((0)\ (0,1))$ gradient path, $K_\varphi = (0,1)$

$q_\varphi = (0)$

$\gamma(\varphi) = -1, \quad m(\varphi) = -1$

φ: is a connection from $\overset{a'}{(1)}$ and $\overset{a}{(0,2)}$

∂ : $\quad a' < K_\varphi \quad q_\varphi < a$

$$\tilde{\partial}((0,2),(1)) = \underbrace{\partial((0,2),(1))}_{(1)} + \underbrace{\sum_{(0,2) \xrightarrow{\varphi} (1)} m(\varphi)}_{-1}$$

$\tilde{\partial}_2\ (0,1,2) \mapsto (1,2) - (0,2)$

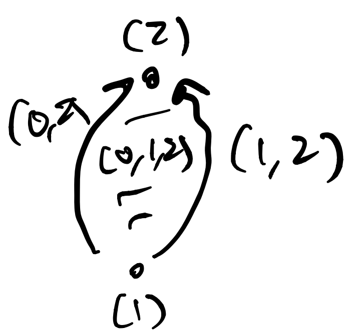
$\tilde{\partial}_1 : (1,2) \mapsto (2) - (1)$

$\quad (0,2) \mapsto (2) - (1)$

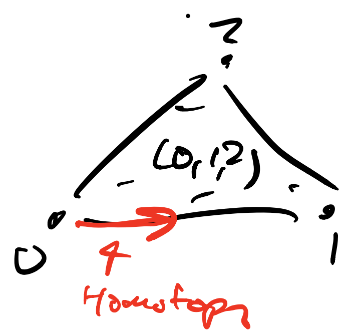
$\tilde{\partial}_0 : (1) \mapsto 0$

$\quad (2) \mapsto 0$

Corresponding Morse Complex



$\simeq$



Essentially, we performed a refraction of the edge $(0,1)$ to the vertex (1)

Pf of thm $H_*(X) \cong H_*(A)$ we proceed by looking at a single refraction at a time

let $q \in X$, $w: q \mapsto w(q)$ $\underset{A}{\underset{\shortparallel}{X_q}} = X \setminus \{q, w(q)\}$

$k_q : X_q \times X_q \to \mathbb{R}$

$$k_q(x,y) = k(x,y) - \frac{k(x,q) \cdot k(w(q),y)}{k(w(q),q)}$$

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define $F_*: C_*(X) \to C_*(X_q)$ by

$$F(x) = \begin{cases} 0 & \text{if } x = w(q) \\ -\sum_{y \in X_q} \frac{k(w(q),y)}{k(w(q),q)}\, y & \text{if } x = q \\ x & \text{otherwise} \end{cases}$$

define $G_*: C_*(X_q) \to C_*(X)$

$$G(x) = x - \frac{k(x,q)}{k(w(q),q)}\, w(q).$$

Exercise: both F, G are chain maps.

Lemma: F and G are chain equivalences i.e. are maps st. $G \circ F \simeq id$, $F \circ G \simeq id$

Pf: $F \circ G = id^{C_*(X_q)}$ (check)

$G \circ F \simeq id^{C_*(X)}$ chain htpy

$$H_n(x) = \begin{cases} \frac{1}{k(w(q),q)} w(q) & \text{if } x = q \\ 0 & \text{otherwise} \end{cases}$$

This is htpy of q collapsing through $w(q)$.

Can check: $H_{n-1}\partial_n + \partial_{n+1} H_n = id - G \circ F$

Finally, want to check that we can chain these operations together. i.e. can go through 1 collapse at a time to obtain Morse complex.

Sketch: collapsing a single pair $(q, w(q))$ doesn't affect rest of vector field and $\bar{K}_q$ are consistent

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M + N 2013: extend this to filtrations using def of filtered acyclic matching on filtration $\{X^t\}_{t \in T}$

$(A^t, w^t : Q^t \to K^t)$ for each X^t filtered

$A^t \subseteq A^{t+1}$, $Q^t \subseteq Q^{t+1}$, $K^t \subseteq K^{t+1}$, $w^t = w^{t+1}|_{Q^t}$ $\forall t$

this gives a filtration on Morse cpx.

$M : (A^t, K^t)$

Furthermore, homological equivalence at every step ensures $PH(X^t) = PH(M^t)$ (up to 0-length)

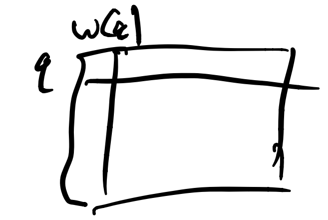
Schur Complements

Let's revisit the formula for K_q

$$K_q(x,y) = K(x,y) - \frac{K(x,q)\,K(w(q),y)}{K(w(q),q)}$$

assume that K derived from ∂, we'll permute $q, w(q)$ to first row/col respectively.

∂ :



recall the Schur complement formula. This shows up in blocked Gaussian elimination.

Let $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, with A_{11} invertible

$$\text{Then, } A = \overset{L}{\begin{bmatrix} I & 0 \\ A_{21}A_{11}^{-1} & I \end{bmatrix}} \begin{bmatrix} A_{11} & 0 \\ 0 & S \end{bmatrix} \overset{U}{\begin{bmatrix} I & A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix}}$$

S is the "Schur complement"

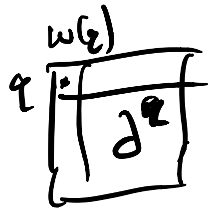
$$\begin{bmatrix} A_{11} & 0 \\ A_{21} & S \end{bmatrix} \begin{bmatrix} I & A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix}$$

$$\|$$

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & S + A_{21}A_{11}^{-1}A_{12} \end{bmatrix}$$

$$\Rightarrow S = A_{22} - A_{21}A_{11}^{-1}A_{12}$$

Let's look a Schur complement of



$$\partial^q[i,j] = \partial[i,j] - \frac{\partial[i, w(q)]\,\partial[q,j]}{\partial[q, w(q)]}$$

compare to

$$k_q(x,y) = k(x,y) - \frac{k(x,q)\,k(w(q),y)}{k(w(q),q)}$$

Same form

Lemma from Numerical lin alg

Suppose $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ with A_{11} invertible.

Then Schur complement $S = A_{22} - A_{21}A_{11}^{-1}A_{12}$ is same whether we use block inverse A_{11}^{-1} or we compute using recursion on invertible submatrices of A_{11}

Schur complement viewpoint on Morse complex
discrete Morse vector field identifies
an invertible block of ∂_k. Boundary $\tilde{\partial}$
is Schur complement wrt. these blocks.
w/ columns removed (clearing)
rows removed (compression)
<u>if</u> they are matched in different dimensions.

invertible block identifies basis
for subset of rng ∂_k
Schur complement just ensures new basis
is independent of this subset.

Final notes:

- Perseus software. demonstrated speedups
- pairs $(q, w(q))$ are persistence pairs.
- speedups can be interpreted as blocking alg w/ optimizations.
- Ripser & Eirene also use for pre-processing
- Eirene (Henselman)