Today:
Klein Bottles
Torsion
Image Patches

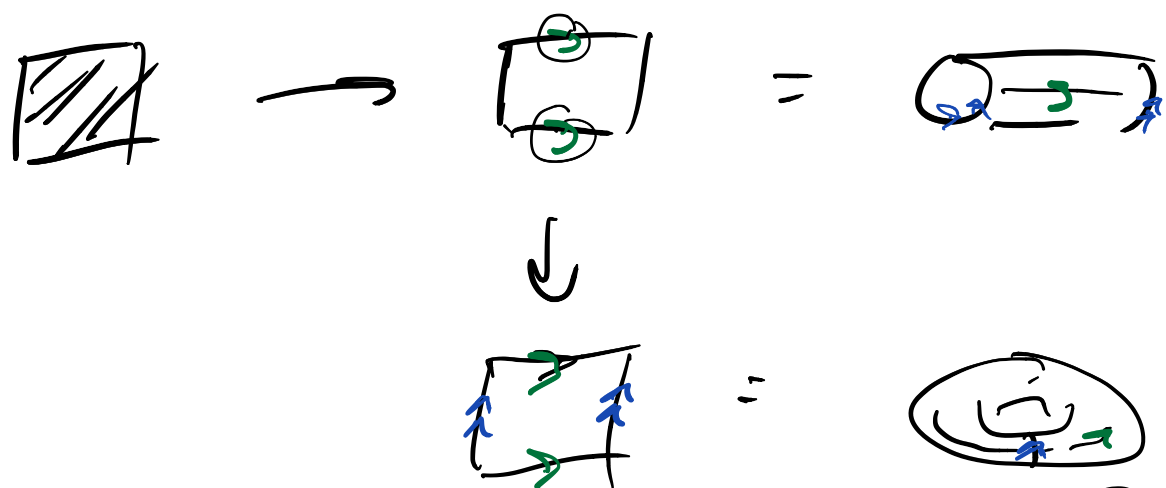
Reminder:
HW 2 is due Fri
HW 1 graded
Project proposals commented

Ref: Carlsson et al "On the local behavior of spaces of image patches" 2008

Influential result.

1) What is a klein bottle

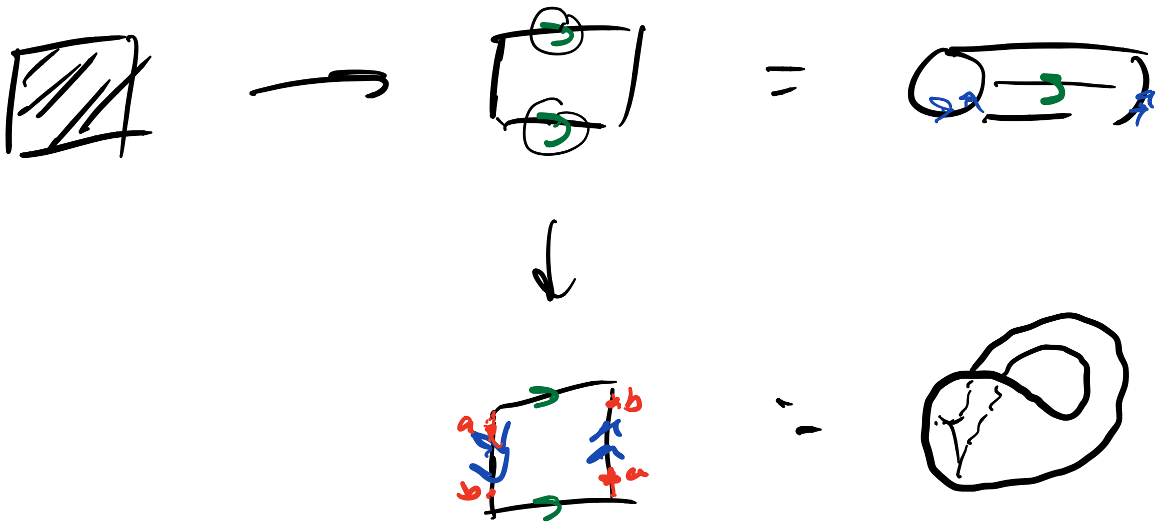
Paper model / identification space



Cylinder

Torus

T^2

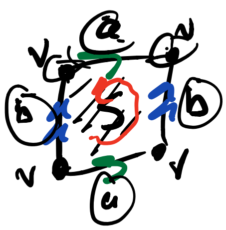


Cylinder

Klein bottle

(can't embed in $\mathbb{R}^3$
need at least $\mathbb{R}^4$)

For computations by hand, often easier to use cell complexes

T^2:  :

$C_2: \langle S \rangle$

$C_1: \langle a, b \rangle$

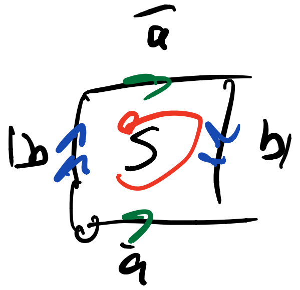
$C_0: \langle v \rangle$

$\partial_2: S \mapsto \cancel{a} + \cancel{b} - \cancel{a} - \cancel{b} = 0$

$\partial_1: \begin{cases} a \longmapsto v - v = 0 \\ b \longmapsto v - v = 0 \end{cases}$

$\partial_0: v \longmapsto 0$

$H_k(T^2) = \begin{cases} 0 & k \geq 2 \\ \mathbb{F} & k = 2 \\ \mathbb{F}^2 & k = 1 \\ \mathbb{F} & k = 0 \end{cases}$



$\partial_2 : S \mapsto a - b - a - b = -2b$

$\partial_1 :$ $a \mapsto v - v = 0$
$b \mapsto v - v = 0$

$\partial_0 : v \mapsto 0$

Fun fact: Homology will depend on what field we use! This is due to "torsion" (ie twisting)

For complete account we would need homology w/ integer coeffs. (not a field)

See Munkres or Hatcher

We'll see how computations play out.

in $\mathbb{F}_2$ coeffs: $\mathbb{Z}/2\mathbb{Z} : \{0, 1\}$
$\rightsquigarrow 2 \equiv 0 \mod 2$

$\partial_2 : S \mapsto 0 \quad (-2b = 0b \mod 2)$

$$H_i(K_4; \mathbb{F}_2) = \begin{cases} \mathbb{F}_2 & i = 2 \\ \mathbb{F}_2^2 & i = 1 \\ \mathbb{F}_2 & i = 0 \end{cases}$$

Klein bottle

Just like the torus.
(Homology w/ $\mathbb{F}_2$ coeffs can't tell apart K and T^2)

in $\mathbb{F}_3$ coeffs: $\mathbb{Z}/3\mathbb{Z}$: $-2 = 1 \bmod 3$.

so $\partial_2 : S \longmapsto \underline{\underline{b}}$

$\ker \partial_2 = \emptyset$ $\quad H_2(K;\mathbb{F}_3) \mid \ker \partial_2 / \underset{0}{\underline{\operatorname{Img} \partial_3}} = 0$

$\ker \partial_1 / \operatorname{Img} \partial_2 = \langle a, b \rangle / \langle b \rangle = \langle a \rangle \cong \mathbb{F}_3$

$H_1(K; \mathbb{F}_3) = \mathbb{F}_3$

$\ker \partial_0 / \operatorname{Img} \partial_1 = \langle v \rangle / 0 = \langle v \rangle \cong \mathbb{F}_3$

This computation will be the same in any field where "2" is invertible (has mult. inverse)

$\mathbb{F}_3$ $2 \cdot 2 = 4 = 1 \bmod 3$

$\mathbb{F}_5$: $2 \cdot 3 = 6 = 1 \bmod 5$

$\mathbb{Q}$: $2 \cdot \frac{1}{2} = 1$

$$H_i(K;\mathbb{F}) = \begin{cases} 0 & i > 1 \\ \mathbb{F} & i = 1 \\ \mathbb{F} & i = 0 \end{cases}$$

not the same as w/ $\mathbb{F}_2$ coeffs

not the same as $H_i(T^2;\mathbb{F})$

$\Rightarrow$ can tell apart K, T^2 w/ $\mathbb{F}_3$ coeffs.

The fact that $\mathbb{F}_2$ coeffs is different indicates that K has "2-torsion" (there can also be 3-torsion, etc.)
This is related to the fact that K is not orientable.

In case you see integer coeffs:

$$H_k(K;\mathbb{Z}) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & k=1 \\ \mathbb{Z} & k=0 \end{cases}$$

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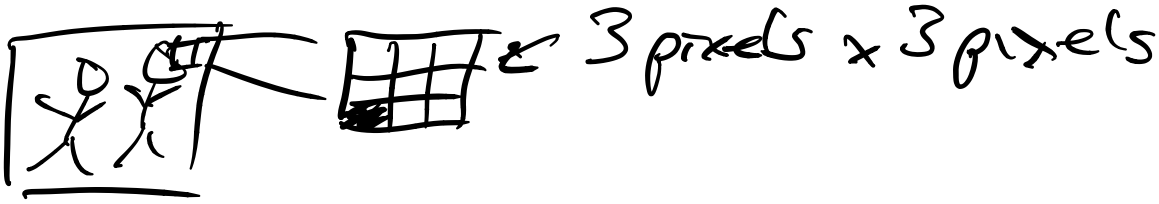
why use different field coeffs in PH?
$\mathbb{F}_2$ can't tell apart K and T,
but $\mathbb{F}_3$ can.

Klein bottle in image patches.

Data: 3x3 image patches



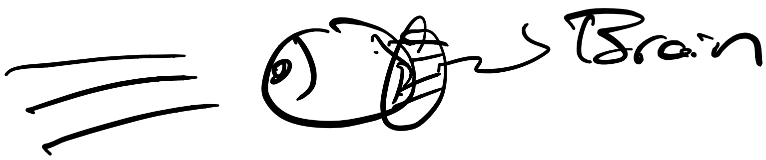
Patches are sampled from Van Hateren database (no compression artifacts, Greyscale)

Natural images: images of everyday scenes

- not random noise
- not digitally generated
- trees, buildings, etc.

Lee, Pedersen, Mumford 2003: Look at these patches to understand local structure.

Related to work on sensitivities in the visual cortex. Tanaka '97: Klein bottle in visual cortex sensitivities.



Relevant: Gabor filters

Result of LPM: there is an annulus (thickened circle) in the data

de Silva & Carlsson used this data set in witness cpx paper. Found "3-circle model"
2004

Carlsson et al 2008: Klein bottle model, includes 3-circle model in 1-skeleton.

Data pre-processing:

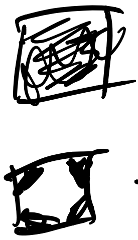
LPM: Take patches from VH dataset.

1) Take log pixel intensity. of images
2) Sample 3x3 patches
3) mean-center $x \mapsto x - \frac{1}{9}\sum_{i,j} x_{ij}$
4) Take top $q\%$ by contrast norm.

essentially: $\|(\nabla_x p, \nabla_y p)\|$

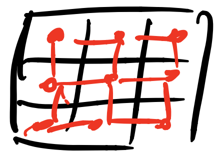
$x^T L x$

graph Laplacian



→ low contrast

→ high contrast

on $G =$ 

nullspace of L:

$x = 1$

5) normalize patches $x \mapsto x/\|x\|$

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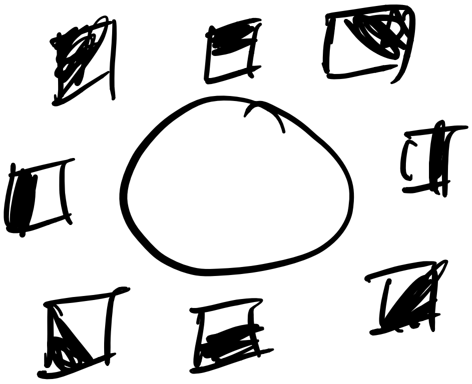
de Silva, Carlsson:

6) compute distances to k^{th} nearest nbr
7) Filter by top $p\%$ to knn.

"codensity filter"

want to look at high-density subsets of data q often 20% (contrast)

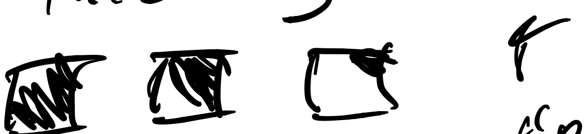
LPM annulus: Edges at different orientations



Why is this important feature?

1) we use contrast <u>for</u> selection.

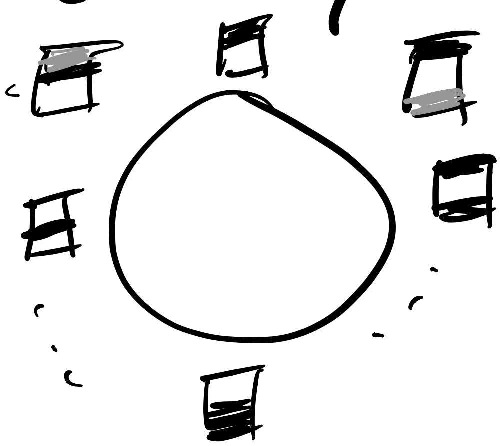
Thickening:

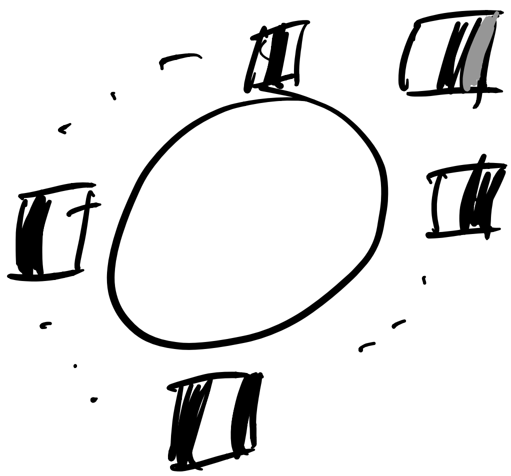


2) These edges are codimension 1.

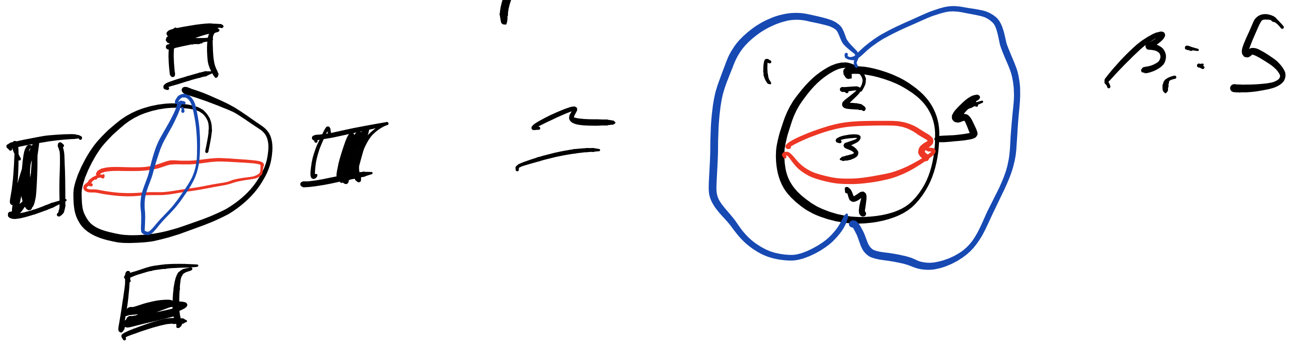
"primary circle"

de Silva & Carlsson: depending on k, p, we see primary circle, or also additional secondary circles.



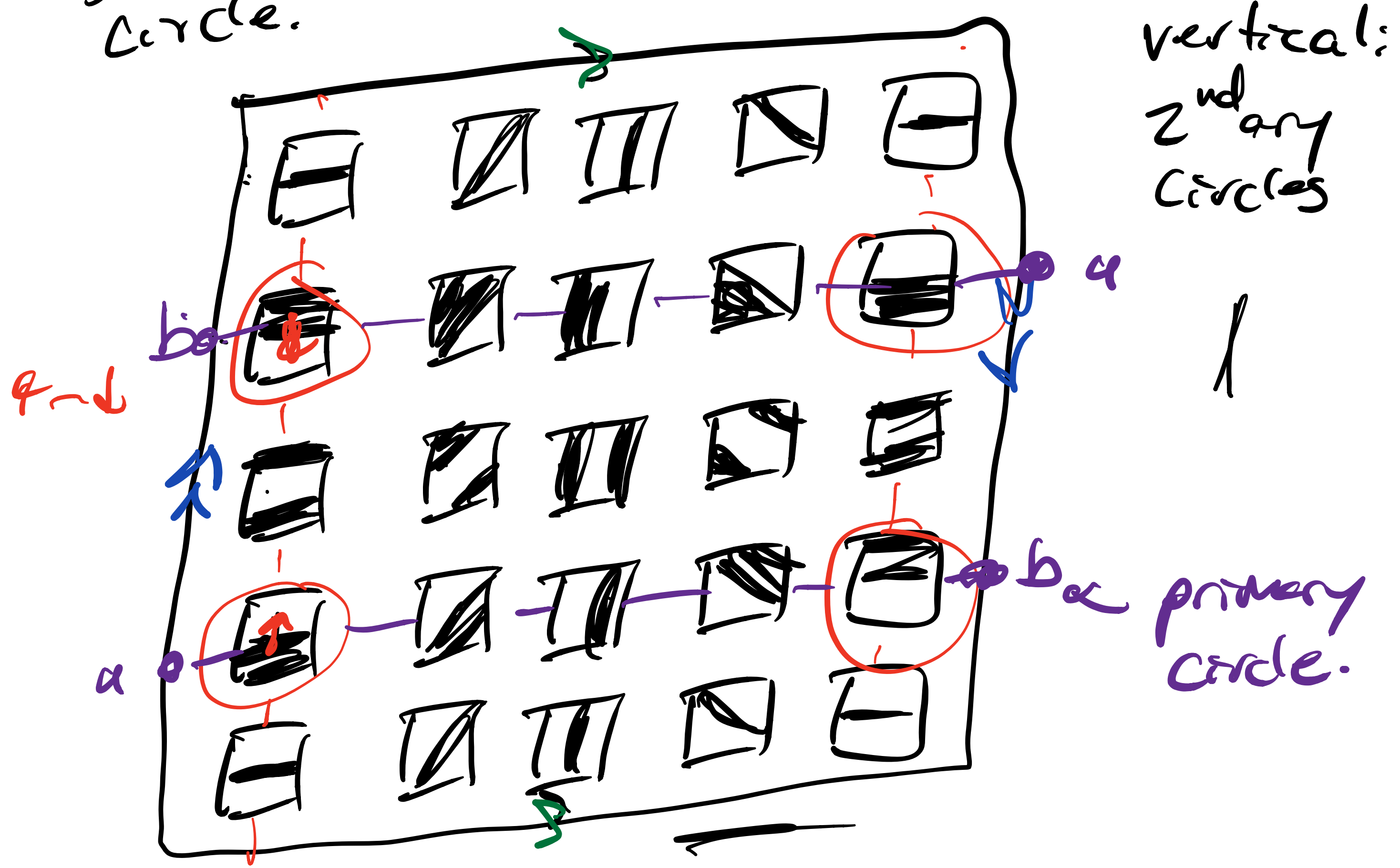


Three circle model: Primary circle + 2 secondary circles:



These 2 secondary circles: appear b/c natural images tend to be aligned w/ buildings windows, which have horizontal + vertical edges.

Carlsson et al 2008: klein bottle model: give every orientation of edge a secondary circle.



How do they justify klein bottle model.

1) Generate a model M (sampling)

$k(\vec{x}\,;\theta,\varphi) : \cos\theta\,(x^T v_\varphi)^2 + \sin\theta\,(x^T v_\varphi)$

v_φ: unit vector on S^1: $\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix}$ $\varphi\in(0,2\pi)$ → location on primary circle

$\theta\in(0,2\pi)$

↳ phase on secondary circle

2) Show data (filter by k, p) union w/ model has homology of model.

Applications:

1) Image compression dictionary
2) Rotation invariant texture recognition (Perea)
3) Extensions to neural networks.

View of this klein bottle as a fiber bundle (twisted product space)

Nelson '20: There's a fibration structure on klein bottle in image patches that generalizes to higher dimensions.

What are components of product?

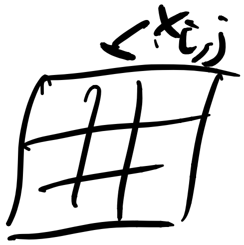
Secondary circles $\times$ (half) the primary circle.

There is a map: $h: \underset{\text{patches}}{X} \to \underset{\text{primary circle}}{S^1}$

that is a <u>fibration</u>

i.e. inverse images are secondary circles.

based on Harris edge/corner detector.

X: 

$$M(x) = \sum_{i,j} \Delta(x_{i,j})\,\Delta(x_{i,j})^T$$

$$\Delta(x_{i,j}) = \begin{pmatrix} x_{i,j+1} - x_{i,j} \\ x_{i+1,j} - x_{i,j} \end{pmatrix}$$

"finite difference gradient"

$$h: x \mapsto \text{MaxEigVect}(M(x))$$

measures direction of largest variation in a patch.

Eigenvectors have sign/<u>scale</u> ambiguity

$$h: \mathbb{R}^9 \to \underline{\mathbb{RP}}^1 \simeq S^1$$